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# On Certain Orthogonal Polynomials, Nikolski- and Turán-Type Inequalities, and Interpolatory Properties of Best Approximants

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For $f \in C[-1, 1]$ denote by $B_{n,p}(f)$ its best $L_p$-approximant by polynomials of degree at most $n$ $(1 \leqslant p \leqslant \infty)$. The following statement is the main result of the paper: Let $1 < p \leqslant \infty$, $f \in C[-1, 1]$, and assume that for a given $(a, b) \subset [-1, 1]$ there exists a sequence of integers $n_1 < n_2 < \cdots < n_j < \cdots$ such that $f - B_{n_j,p}(f)$ is zero free on $(a, b)$. Then $\limsup_{j \to \infty} n_{j+1}/n_j > 1$. 

## INTRODUCTION AND MAIN RESULTS

For a real continuous function $f(x)$ on $I = [-1, 1]$ denote

$$\|f\|_\infty = \max_{x \in I} |f(x)|, \qquad \|f\|_p = \left( \int_I |f(x)|^p \, \omega(x) \, dx \right)^{1/p}, \qquad 1 \leqslant p < \infty,$$

where $\omega$ is positive a.e. on $I$ and $\int_I \omega \, dx = 1$. Furthermore, $B_{n,p}(f)$ denotes the best $L_p$-approximant on $I$ of function $f(x)$ from the space $P_n$ of algebraic polynomials of degree at most $n$, i.e.,

$$E_{n,p}(f) = \inf_{g \in P_n} \|f - g\|_p = \|f - B_{n,p}(f)\|_p, \qquad 1 \leqslant p \leqslant \infty.$$

It is well known that, for every $1 \leqslant p \leqslant \infty$ and $n \in \mathbf{N}$, $f - B_{n,p}(f)$ has at least $n + 1$ distinct zeros in $I$. Moreover, the following general result on density of zeros of $f - B_{n,p}(f)$ holds.

THEOREM A. *Let $1 \leqslant p \leqslant \infty$, $f \in C[-1, 1]$, and $-1 \leqslant a < b \leqslant 1$. Then there exists a subsequence of integers $\{n_j\}_{j=1}^{\infty}$, $n_1 < n_2 < \cdots$, such that $f - B_{n_j,p}(f)$ vanishes on $[a, b]$ $(j \in \mathbf{N})$.*

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The first step in verifying the above statement was done by Kadec [1], who proved it for $p=\infty$ (in a stronger form). For $p=1$ the above theorem first appears in Kroó and Peherstorfer [2]. It is important to point out that for $p=1$, Theorem A holds for every $n \geqslant n_0$ and not just a subsequence of $n$'s.

Subsequently, Saff and Shekhtman [7] proved Theorem A for $p=2$, and then it was extended by Kroó and Swetits [3] for $1<p<\infty$, $p \neq 2$. Thus the density of zeros of $f-B_{n,p}(f)$ was shown to hold for a subsequence of $n$'s if $1<p\leqslant\infty$, while for $p=1$ it was verified for every $n$ large enough. Naturally, this leads to the question whether Theorem A can hold for every sufficiently big $n$ when $1<p\leqslant\infty$. The next result gives a negative answer to this question. It was verified by Lorentz [4] for $p=\infty$, Saff and Shekhtman [7] for $p=2$, and Kroó and Swetits [3] for $1<p<\infty$, $p\neq 2$.

THEOREM B. *Let* $1<p\leqslant\infty$ *and* $-1<a<b<1$. *Then there exists an entire function* $f$ *and a subsequence of integers* $n_1<n_2<\cdots$, *such that* $f-B_{n_j,p}(f)$ *is zero-free on* $[a,b]$ $(j\in\mathbf{N})$.

Thus when $1<p\leqslant\infty$ a function may possess a "bad" subsequence of integers, where the density fails.

In their paper [7] Saff and Shekhtman offered the following conjecture concerning bad subsequences.

*Conjecture.* Let $1<p\leqslant\infty$, $f\in C[-1,1]$, and assume that for a given subsequence $\{n_j\}$, $j\in\mathbf{N}$, $f-B_{n_j,p}(f)$ is zero-free on some interval $(a,b)\subset[-1,1]$. Then $\{n_j\}$ is in some sense lacunary.

The main goal of this paper is to given the following affirmative answer to the above conjecture.

THEOREM 1. *Let* $1<p\leqslant\infty$, $f\in C[-1,1]$, *and assume that for a given subsequence* $\{n_j\}_{j=1}^{\infty}$, $n_1<n_2<\cdots$, $f-B_{n_j,p}(f)$ *is zero-free on some interval* $(a,b)\subset[-1,1]$. *Then* $\limsup_{j\to\infty} n_{j+1}/n_j>1$.

*Remarks.* First let us note that Theorem 1 can be formulated in the following equivalent way: if $\{n_j\}_{j=1}^{\infty}$ is such that $\lim_{j\to\infty} n_{j+1}/n_j=1$ then for any $[a,b]\subset I$ there exists a subsequence $\{n_{j_k}\}_{k=1}^{\infty}$ such that $f-B_{n,p}(f)$ vanishes on $[a,b]$ for $n=n_{j_k}$ $(k\in\mathbf{N})$. In this respect Theorem 1 is a generalization of Theorem A. Let us also point out that Theorem 1 is sharp in the sense that stronger "lacunarity" of subsequence $\{n_j\}_{j=1}^{\infty}$ need not hold in general. This claim is endorsed by examples of zero-free subsequences constructed in Lorentz [4]. Saff and Shekhtman [7] and Kroó and Swetits [3] showed that for these subsequences $\limsup_{j\to\infty} n_{j+1}/n_j$ can be arbitrarily close to (but bigger than) one, and $\liminf_{j\to\infty} n_{j+1}/n_j$

may equal one. Finally, we mention that all results presented above can be given in slightly more general form by replacing the study of zeros by oscillation points ($p=\infty$) or points of sign change ($1 \leqslant p < \infty$). Since the proofs remain identical we decided to present the more attractive case of zero distribution.

The proof of Theorem 1 will be divided into three cases: $p=2$; $p=\infty$; $1<p<\infty$, $p \neq 2$. Each of these three components of proof requires a somewhat different technique, but the main difficulties and ideas of the proof are concentrated in the case when $p=2$. The case $p=2$ is of a special interest also in view of the fact that $L_2$-versions of the above results correspond to the zero distribution of remainders of Fourier series. The proof of Theorem 1 for $p=2$ will be based on certain general results concerning orthogonal polynomials which appears to be of independent interest. In order to formulate this result we introduce some additional notations. We say that $f \in L_2[-1, 1]$ is orthogonal to $P_n$, written $f \perp P_n$, if for every $g_n \in P_n$

$$\int_{-1}^{1} f g_n \omega \, dx = 0.$$

Furthermore, for $f \in C[-1, 1]$, $N(f, [a, b])$ stands for the number of zeros of $f$ on $[a, b] \subset [-1, 1]$. Now we present the main auxiliary result of the paper which turns out to be crucial for verifying Theorem 1.

THEOREM 2. *Let $g_{m_n}$, $\|g_{m_n}\|_2 = 1$ be a sequence of polynomials of degree at most $m_n$ orthogonal to $P_n$ ($n < m_n$, $n \in \mathbb{N}$), and assume that $m_n/n \to 1$ as $n \to \infty$. Then for every $[a, b] \subset [-1, 1]$ the following two relations hold*:

$$(1) \quad \lim_{n \to \infty} \frac{1}{n} N(g_{m_n}, [a, b]) = \frac{1}{\pi}(\arccos a - \arccos b)$$

$$(2) \quad \lim_{n \to \infty} \left( \int_a^b g_{m_n}^2 \omega \, dx \right)^{1/n} = 1.$$

*Remarks.* When $m_n = n+1$, i.e., $g_{m_n} = g_{n+1}$ is the orthonormal polynomial related to the weight $\omega$, the uniform distributions of zeros of $g_{m_n}$ (see (1)) is well known. Moreover, for $m_n = n+1$ relation (2) is just a consequence of Turán's inequality (see [8, 5]). Theorem 2 extends properties (1)–(2) of "orthogonal" polynomials $g_{m_n}$ to the case when $m_n = n + o(n)$. On the other hand if $\lim_{n \to \infty} m_n/n > 1$ then (1) and (2) fail to hold, in general. This can be seen by considering weight $\omega = 1$ and incomplete polynomials of the form $g_{m_n}(x) = (x+1)^{m_n - n - 1} \tilde{g}_{n+1}(x)$ where $\tilde{g}_{n+1} \in P_{n+1}$ is chosen so that $\|g_{m_n}\|_2 = 1$ and $g_{m_n} \perp P_n$. If $\lim_{n \to \infty} m_n/n > 1$ then for sufficiently small $\delta > 0$ interval $(-1, -1+\delta)$ does not contain zeros of $g_{m_n}$ (see [6]) and $\int_{-1}^{-1+\delta} g_{m_n}^2 \, dx$ tends to zero geometrically as $n \to \infty$. Thus the condition $m_n = n + o(n)$ is necessary in order that (1) and (2) hold.

## Properties of Certain Orthogonal Polynomials, Nikolski- and Turán-Type Inequalities and Proof of Theorem 2

First we need certain Nikolski-type inequalities, which estimate from above the $L_\infty$-norm of a polynomial by its $L_p$-norm. For the positive a.e. weight $\omega$ satisfying $\int_{-1}^{1} \omega \, dx = 1$ set $\varphi(\omega, \varepsilon) = \inf\{\int_A \omega \, dx : A \subset [-1, 1], \mu(A) \geq \varepsilon\}$, $0 \leq \varepsilon \leq \pi$, where $\mu(A) = \int_A dx/\sqrt{1-x^2}$ is the Chebyshev measure of $A$. Furthermore, let $\varepsilon_n(\omega)$ be the unique solution of the equation $\varphi(\omega, \varepsilon) = e^{-n\varepsilon}$. Then $\varepsilon_n(\omega) \to 0$ as $n \to \infty$. It is shown in [3] that for every $g_n \in P_n$ and $0 < p \leq \infty$

$$\|g_n\|_\infty \leq e^{cn\varepsilon_n(\omega)} \|g_n\|_p, \tag{3}$$

where $c > 0$ depends only on $p$ and $\omega$.

In particular, (3) implies that

$$\lim_{n \to \infty} \left\{ \sup_{g_n \in P_n \setminus \{0\}} \frac{\|g_n\|_\infty}{\|g_n\|_p} \right\}^{1/n} = 1. \tag{4}$$

Now we want to address the question whether (4) can hold if the weight $\omega$ is varying with $n$, That is, we consider a sequence of weights $\omega_n$ ($n \in \mathbf{N}$, $\omega_n > 0$ a.e., $\int_{-1}^{1} \omega_n \, dx = 1$) and the corresponding $L_{p,n}$-norms $\|f\|_{p,n} = (\int_{-1}^{1} |f|^p \omega_n \, dx)^{1/p}$. It turns out that (4) remains true with $L_{p,n}$-norms provided that $\omega_n^{1/n} \to 1$ in measure, that is, for any $\varepsilon, \delta > 0$, $\mu_0\{|\omega^{1/n} - 1| > \delta\} < \varepsilon$ whenever $n \geq n_0(\varepsilon, \delta)$. Here and in what follows $\mu_0(\ldots)$ stands for the Lebesgue measure.

LEMMA 1. *Let $\omega_n$ ($n \in \mathbf{N}$, $\omega_n > 0$ a.e., $\int_{-1}^{1} \omega_n \, dx = 1$) be a sequence of weights such that $\omega_n^{1/n} \to 1$ in measure. Then for every $1 \leq p < \infty$*

$$\lim_{n \to \infty} \left\{ \sup_{g_n \in P_n \setminus \{0\}} \frac{\|g_n\|_\infty}{\|g_n\|_{p,n}} \right\}^{1/n} = 1. \tag{5}$$

*Proof.* First we need a more precise version of (3) with $c$ independent of $\omega$. In case 1 of Lemma 2, in [3] it is shown that if $\varepsilon_n(\omega_n)$ is the solution of equation $\varphi(\omega_n, \varepsilon) = e^{-n\varepsilon}$, and for a given $n \in \mathbf{N}$, $\varepsilon_n(\omega_n) < \pi/4$ then for every $g_n \in P_n$ and $1 \leq p < \infty$

$$\|g_n\|_\infty \leq e^{c_1 n\varepsilon_n(\omega_n)} \|g_n\|_{p,n} \tag{6}$$

with an absolute constant $c_1 > 0$. Since $\omega_n^{1/n} \to 1$ in measure (thus also in Chebyshev measure) as $n \to \infty$ there exists an $n_0 \in \mathbf{N}$ such that whenever $n \geq n_0$

$$\mu\{|\omega_n^{1/n} - 1| > 1 - e^{-\pi/5}\} < \frac{\pi}{8},$$

i.e., $\mu\{\omega_n < e^{-\pi n/5}\} < \pi/8$. Thus if $n \geq n_0$

$$\begin{aligned}\varphi\left(\omega_n, \frac{\pi}{4}\right) &= \inf\left\{\int_A \omega_n\, dx : \mu(A) \geq \frac{\pi}{4}\right\}\\ &\geq e^{-\pi n/5} \inf\left\{\int_B dx : \mu(B) \geq \frac{\pi}{8}\right\} \geq \left(1 - \frac{\sqrt{3}}{2}\right) e^{-\pi n/5}.\end{aligned}$$

On the other hand if $\varepsilon_n(\omega_n) \geq \pi/4$ for some $n \geq n_0$ then

$$e^{-\pi n/4} \geq e^{-n\varepsilon_n} = \varphi(\omega_n, \varepsilon_n) \geq \varphi\left(\omega_n, \frac{\pi}{4}\right) \geq \left(1 - \frac{\sqrt{3}}{2}\right) e^{-\pi n/5},$$

i.e., $n \leq n_1 = [(20/\pi)\ln(2/(2-\sqrt{3}))]$. Thus $\varepsilon_n(\omega_n) < \pi/4$ whenever $n > n^* = \max\{n_0, n_1\}$. Hence (6) holds for every $n > n^*$ and $g_n \in P_n$. Now in order to prove (5) it remains to show that $\varepsilon_n(\omega_n) \to 0$ as $n \to \infty$. As above it is easy to show that for any $\varepsilon > 0$ and $\delta > 0$ there exists an $\tilde{n} = \tilde{n}(\varepsilon, \delta)$ such that whenever $n \geq \tilde{n}$, $\varphi(\omega_n, \varepsilon) \geq (1-\delta)^n \tilde{c}(\varepsilon)$ with $\tilde{c}(\varepsilon) > 0$ depending only on $\varepsilon$. This means that $\liminf_{n\to\infty} \varphi(\omega_n, \varepsilon)^{1/n} \geq 1$ for every $\varepsilon > 0$. Assume now, that $\varepsilon_n(\omega_n) \nrightarrow 0$ as $n \to \infty$, that is, $\varepsilon_n(\omega_n) \geq \tilde{\varepsilon} > 0$ for $n \in \Omega$, where $\Omega \subset \mathbf{N}$ is infinite. Then

$$e^{-n\tilde{\varepsilon}} \geq e^{-n\varepsilon_n} = \varphi(\omega_n, \varepsilon_n) \geq \varphi(\omega_n, \tilde{\varepsilon}), \qquad n \in \Omega.$$

Hence $\liminf_{n\to\infty} \varphi(\omega_n, \tilde{\varepsilon})^{1/n} < 1$, a contradiction. Thus $\varepsilon_n(\omega_n) \to 0$ as $n \to \infty$. ∎

The above lemma will be applied for special weights of the form $\omega_n = \omega |g_{r_n}|^\alpha$ where $\omega$ is a fixed weight, $\alpha > 0$, and $g_{r_n}$ is a monic polynomial of degree $r_n = o(n)$ $(n \in \mathbf{N})$ with all its zeros belonging to $[-1, 1]$.

LEMMA 2. *Let $\omega_n = \omega |g_{r_n}|^\alpha$, where $\omega > 0$ a.e. on $I$, $\int_I \omega\, dx = 1$, $\alpha > 0$, and $g_{r_n}$ is a monic polynomial of degree $r_n = o(n)$ $(n, r_n \in \mathbf{N})$ having all its zeros in $I$. Then* (5) *holds for every* $1 \leq p < \infty$.

*Proof.* First we need to verify that $\omega_n^{1/n} \to 1$ in measure as $n \to \infty$. For this end we prove that for every $0 < \gamma < 1$ and $\tilde{g}_n \in P_n$

$$\mu_0\{x \in I : |\tilde{g}_n(x)| \leq \gamma^n \|\tilde{g}_n\|_\infty\} \leq 4e\gamma \qquad (n \in \mathbf{N}). \tag{7}$$

Set

$$A(\gamma)=\{x\in I: |\tilde{g}_n(x)|\leqslant\gamma^n\|\tilde{g}_n\|_\infty\},\qquad \rho=\mu_0(A(\gamma)).$$

Obviously, there exist $x_1, ..., x_{n+1}\in A(\gamma)$ such that $x_{i+1}-x_i\geqslant\rho/n$. Then

$$\tilde{g}_n(x)=\sum_{i=1}^{n+1}\tilde{g}_n(x_i)\,l_i(x), \tag{8}$$

where $l_i(x)=\omega(x)/(x-x_i)\,\omega'(x_i)$, $\omega(x)=\prod_{i=1}^{n+1}(x-x_i)$. We have using that $n!>n^ne^{-n}$

$$\begin{aligned}\|l_i\|_\infty&\leqslant\frac{2^n}{(\rho/n)^n\,(i-1)!\,(n+1-i)!}\\&=\frac{2^n\cdot n^n}{\rho^n\cdot n!}\binom{i-1}{n}\leqslant\left(\frac{2e}{\rho}\right)^n\binom{i-1}{n}.\end{aligned} \tag{9}$$

By (8) and (9)

$$\|\tilde{g}_n\|_\infty\leqslant\sum_{i=1}^{n+1}|\tilde{g}_n(x_i)|\,\|l_i\|_\infty\leqslant\gamma^n\|\tilde{g}_n\|_\infty\left(\frac{4e}{\rho}\right)^n,$$

i.e. $\rho\leqslant 4e\gamma$ and (7) holds.

Let us prove now that $\omega_n^{1/n}=\omega^{1/n}|g_{r_n}|^{\alpha/n}\to 1$ in measure. Since all zeros of $g_{r_n}$ lie in $I$ we have $2^{-r_n+1}\leqslant\|g_{r_n}\|_\infty\leqslant 2^{r_n}$. Hence using that $r_n=o(n)$ we have $\|g_{r_n}\|_\infty^{\alpha/n}\to 1$ $(n\to\infty)$ and

$$0<m\leqslant\|g_{r_n}\|_\infty^{\alpha/n}\leqslant M<\infty\qquad(n\in\mathbf{N}).$$

Since $\omega^{1/n}\to 1$ a.e. (and thus in measure too) and functions $|g_{r_n}|^{\alpha/n}$ are uniformly bounded it suffices to show that $|g_{r_n}|^{\alpha/n}\to 1$ in measure. By (7) for any $0<\gamma<1$

$$\mu_0\{x\in I: |g_{r_n}(x)|^{\alpha/n}\leqslant\gamma^{\alpha r_n/n}\|g_{r_n}\|_\infty^{\alpha/n}\}\leqslant 4e\gamma. \tag{10}$$

Since $\gamma^{\alpha r_n/n}\|g_{r_n}\|_\infty^{\alpha/n}\to 1$ as $n\to\infty$, for any $\varepsilon$, $\delta>0$, setting $\gamma_\varepsilon=\varepsilon/4e$ we have $\gamma_\varepsilon^{\alpha r_n/n}\|g_{r_n}\|_\infty^{\alpha/n}>1-\delta$ and $\|g_{r_n}\|_\infty^{\alpha/n}<1+\delta$ if $n\geqslant n^*(\varepsilon,\delta)$.

Therefore for $n\geqslant n^*(\varepsilon,\delta)$ it follows from (10)

$$\begin{aligned}\mu_0\{||g_{r_n}|^{\alpha/n}-1|>\delta\}&=\mu_0\{|g_{r_n}|^{\alpha/n}>1+\delta\}+\mu_0\{|g_{r_n}|^{\alpha/n}<1-\delta\}\\&=\mu_0\{|g_{r_n}|^{\alpha/n}<1-\delta\}\\&\leqslant\mu_0\{|g_{r_n}|^{\alpha/n}\leqslant\gamma_\varepsilon^{\alpha r_n/n}\|g_{r_n}\|_\infty^{\alpha/n}\}\leqslant 4e\gamma_\varepsilon=\varepsilon.\end{aligned}$$

Thus $|g_{r_n}|^{\alpha/n}\to 1$ in measure, and consequently the same is true for $\omega^{1/n}|g_{r_n}|^{\alpha/n}=\omega_n^{1/n}$.

In order to apply Lemma 1 to weights $\omega_n$ we need to normalize them. Set $\tilde{\omega}_n = \omega_n / \int_I \omega_n \, dx$. By (3)

$$\int_I \omega_n \, dx = \|g_{r_n}\|_\alpha^\alpha \geq e^{-c\alpha r_n \varepsilon_{r_n}(\omega)} \|g_{r_n}\|_\infty^\alpha \geq e^{-c\alpha r_n \varepsilon_{r_n}(\omega)} 2^{(-r_n+1)\alpha}.$$

On the other hand $\int_I \omega_n \, dx \leq \|g_{r_n}\|_\infty^\alpha \leq 2^{r_n \alpha}$. Since $r_n = o(n)$ it follows that

$$\left(\int_I \omega_n \, dx\right)^{1/n} \to 1, \qquad n \to \infty. \tag{11}$$

Furthermore, $\omega_n^{1/n} \to 1$ in measure. Hence and by (11), $\tilde{\omega}_n^{1/n} \to 1$ in measure. Thus by Lemma 1 relation (5) holds for $\tilde{\omega}_n$. Since $\tilde{\omega}_n$ differs from $\omega_n$ only by the constant multiplier $\int_{-1}^1 \omega_n \, dx$ satisfying (11), it easily follows that (5) holds for $\omega_n$, as well. ∎

Let $f \in C[-1, 1]$, $-1 \leq x_1 < \cdots < x_k \leq 1$ be such that $f(x_i) f(x_{i+1}) < 0$ $(1 \leq i \leq k-1)$ and set $\eta = \min_{1 \leq i \leq k} |f(x_i)|$. Then we say that $\{x_i\}_{i=1}^k$ is an oscillation of function $f$ of length $k$ and magnitude $\eta$. In order to prepare for the proof of Theorem 2 we need several technical lemmas concerning oscillatory properties of polynomials.

LEMMA 3. *Assume that* $p_m \in P_m$ *changes sign at* $k$ *points* $-1 < x_1 < \cdots < x_k < 1$ $(k \leq m)$. *Then for any* $m-k \leq n \leq m$ *there exists a monic polynomial* $g_r \in P_r$ *with* $r$ *simple zeros in* $(-1, 1)$, $m-n \leq r \leq m-n+1$, *and a closed set* $A \subset [-1, 1]$ *such that*

(i) $p_m/g_r \in P_{m-r}$;

(ii) $g_r \geq 0$ *on* $I \backslash A$;

(iii) $p_m$ *has an oscillation on* $I \backslash A$ *of length* $k+n-m$ *and magnitude* $\eta \geq \max_{x \in A} |p_m(x)|$.

*Proof.* Consider the maximums of $|p_m|$ on intervals $[x_i, x_{i+1}]$ $(0 \leq i \leq k,\ -1 = x_0,\ x_{k+1} = 1)$ and select $A_1 = [x_j, x_{j+1}]$ $(0 \leq j \leq k)$ so that $|p_m|$ has the smallest maximum on $A_1$. Set $r_1 = 1$ (if $j = 0$ or $k$) or 2 (if $1 \leq j \leq k-1$) and

$$g_{r_1}(x) = \begin{cases} (x - x_j)(x - x_{j+1}), & \text{if } 1 \leq j \leq k-1 \\ (x - x_1), & \text{if } j = 0 \\ (x_k - x), & \text{if } j = k. \end{cases}$$

Then $p_m/g_{r_1} \in P_{m-r_1}$, $g_{r_1} \geq 0$ for $x \in [-1, 1] \backslash A_1$, and $p_m$ has an oscillation on $[-1, 1] \backslash A_1$ of length $(k+1) - r_1$ and magnitude $\eta_1 \geq \max_{x \in A_1} |p_m(x)|$. Now consider the function $p_m(1 - \chi(A_1))$ (as usual $\chi(\ldots)$ denotes the characteristic function). This function has $k - r_1$ sign changes at points $\{x_1, ..., x_{j-1}, x_{j+2}, ..., x_k\}$. Consider again the maximums of $|p_m(1-\chi(A_1))|$ on intervals where it is sign preserving and choose the interval $A_2$ with the smallest maximum. Construct $r_2$ and $g_{r_2}$ as above ($r_2 = 1$ or 2). Then again $p_m/g_{r_1} g_{r_2} \in P_{m-r_1-r_2}$, $g_{r_1} g_{r_2} \geqslant 0$ for $x \in [-1, 1] \backslash (A_1 \cup A_2)$ and $p_m$ has an oscillation on $[-1, 1] \backslash (A_1 \cup A_2)$ of length $(k+1)-(r_1+r_2)$ and magnitude $\eta_2 \geqslant \max_{x \in A_1 \cup A_2} |p_m(x)|$. Now, repeating this procedure $s$ times so that $m-n \leqslant \sum_{j=1}^{s} r_j \leqslant m-n+1$, and setting $r = \sum_{j=1}^{s} r_j$, $g_r = \prod_{j=1}^{s} g_{r_j}$, $A = \bigcup_{j=1}^{s} A_j$ we obtain a monic polynomial $g_r \in P_r$ with $r$ simple zeros in $(-1, 1)$ such that $p_m/g_r \in P_{m-r}$, $g_r \geqslant 0$ on $[-1, 1] \backslash A$; $p_m$ has an oscillation on $[-1, 1] \backslash A$ of length $(k+1)-r \geqslant k+1-(m-n+1) = k+n-m$, and magnitude $\eta \geqslant \max_{x \in A} |p_m(x)|$. ∎

The above lemma combined with the Nikolski-type result provided by Lemma 2 leads to an important statement concerning oscillatory properties of certain orthogonal polynomials.

LEMMA 4. *Let $p_{m_n}$ be a sequence of nontrivial polynomials of degree at most $m_n$ orthogonal to $P_n$ ($m_n > n$, $n \in \mathbf{N}$), where $m_n/n \to 1$. Then, for every $n$, $p_{m_n}$ possesses an oscillation of length $2n - m_n = n + o(n)$ and magnitude $\eta_n$ satisfying*

$$\lim_{n \to \infty} \left( \frac{\eta_n}{\| p_{m_n} \|_\infty} \right)^{1/n} = 1. \tag{12}$$

*Proof.* Since $p_{m_n}$ is orthogonal to $P_n$, it has $k_n \geqslant n+1$ sign changes in $(-1, 1)$. Thus by Lemma 3 there exists monic polynomials $g_{r_n} \in P_{r_n}$ with $r_n$ simple zeros in $(-1, 1)$, $m_n - n \leqslant r_n \leqslant m_n - n + 1$, and closed sets $A_n \subset [-1, 1]$ so that (i)–(iii) hold. Thus by (iii), $p_{m_n}$ has a certain oscillation of length $k_n + n - m_n \geqslant 2n - m_n = n + o(n)$ and magnitude $\eta_n$ satisfying (iii) and it remains to verify (12) for $\eta_n$. By (i), $p_{m_n}/g_{r_n} \in P_{m_n - r_n}$, where $m_n - r_n \leqslant n$. Since $p_{m_n} \perp P_n$ we have by (ii)

$$0 = \int_I \frac{p_{m_n}^2}{g_{r_n}} \omega \, dx = \int_{I \backslash A_n} \frac{p_{m_n}^2}{|g_{r_n}|} \omega \, dx + \int_{A_n} \frac{p_{m_n}^2}{g_{r_n}} \omega \, dx.$$

Therefore using (iii)

$$\int_I \frac{p_{m_n}^2}{|g_{r_n}|} \omega \, dx \leqslant 2 \int_{A_n} \frac{p_{m_n}^2}{|g_{r_n}|} \omega \, dx \leqslant 2\eta_n \| p_{m_n}/g_{r_n} \|_\infty . \tag{13}$$

On the other hand setting $\omega_n = \omega |g_{r_n}|$ we have

$$\int_I \frac{p_{m_n}^2}{|g_{r_n}|} \omega \, dx = \int_I |p_{m_n}/g_{r_n}|^2 \, \omega_n \, dx = \| p_{m_n}/g_{r_n} \|_{2,n}^2 \tag{14}$$

where the $\|\cdot\|_{2,n}$ norm corresponds to the weight $\omega_n$, and $p_{m_n}/g_{r_n} \in P_n$. Furthermore, $\lim_{n\to\infty} r_n/n = \lim_{n\to\infty} (m_n/n - 1) = 0$, i.e., weights $\omega_n$ satisfy requirements of Lemma 2. Hence for every $\tilde{g}_n \in P_n$

$$\|\tilde{g}_n\|_\infty \leqslant e^{n\alpha_n} \|\tilde{g}_n\|_{2,n},$$

where $\alpha_n \to 0^+$ $(n \to \infty)$. Using (13), (14), and the above inequality for $\tilde{g}_n = p_{m_n}/g_{r_n}$

$$2\eta_n \|p_{m_n}/g_{r_n}\|_\infty \geqslant \|p_{m_n}/g_{r_n}\|_{2,n}^2 \geqslant e^{-2n\alpha_n} \|p_{m_n}/g_{r_n}\|_\infty^2,$$

i.e.,

$$\eta_n \geqslant \tfrac{1}{2} e^{-2n\alpha_n} \|p_{m_n}/g_{r_n}\|_\infty \geqslant 2^{-r_n-1} e^{-2n\alpha_n} \|p_{m_n}\|_\infty.$$

Since $r_n = o(n)$ and $\alpha_n \to 0^+$ as $n \to \infty$ the last inequality implies (12). ∎

According to Lemma 4 polynomials $p_{m_n}$ possess oscillations of "asymptotically optimal" magnitude and length. Our next lemma shows that points of such oscillations have uniform distribution with respect to the Chebyshev measure.

LEMMA 5. *Let $p_{m_n} \in P_{m_n}$ $(m_n/n \to 1$ as $n \to \infty)$ possess an oscillation of length $n - k_n$ $(k_n = o(n))$ and magnitude $\geqslant \eta_n$ with $\eta_n$ satisfying* (12). *Then for every $[a, b] \subset [-1, 1]$*

$$\lim_{n\to\infty} \frac{1}{n} \tilde{N}(p_{m_n}, [a, b]) = \frac{1}{\pi} (\arccos a - \arccos b),$$

*where $\tilde{N}(p_{m_n}, [a, b])$ stands for the number of oscillation points of $p_{m_n}$ in $[a, b]$.*

*Proof.* Without loss of generality we may assume that $\|p_{m_n}\|_\infty = 1$ (i.e., $\eta_n^{1/n} \to 1$) and $\eta_n \to 0$ (replacing $\eta_n$ by $\min\{\eta_n, 1/n\}$). Then setting $\delta_n = -(1/n) \ln \eta_n \geqslant 0$ we have $\delta_n \to 0$ and $n\delta_n \to \infty$ $(n \to \infty)$. Furthermore, set $s_n = [n\sqrt{\delta_n}]$, $\xi_n = \sqrt[3]{\delta_n}$ $(s_n \to \infty$ and $\xi_n \to 0$ as $n \to \infty)$ and let

$$Q_{m_n + 2s_n}(x) = p_{m_n}(x) + \frac{\eta_n}{2} T_{m_n}(x)\, q_{2s_n}(x),$$

where $T_k(x) = \cos k \arccos x$,

$$q_{2s_n}(x) = \left(\frac{4 - (x - (a+b)/2)^2}{4 - ((a-b)/2)^2}\right)^{s_n}.$$

Note that $|q_{2s_n}(x)| \leqslant 1$ for $x \in [-1, 1] \backslash (a, b)$. Moreover, for $n$ sufficiently large $\xi_n \leqslant (b-a)/4$, and therefore for any $x \in [a+\xi_n, b-\xi_n]$

$$q_{2s_n}(x) \geqslant \left(\frac{4-(a+\xi_n-(a+b)/2)^2}{4-((a-b)/2)^2}\right)^{s_n} = \left(1+\frac{\xi_n(b-a-\xi_n)}{4-((a-b)/2)^2}\right)^{s_n}$$
$$\geqslant \left(1+\frac{1}{8}(b-a)\,\xi_n\right)^{s_n} \geqslant e^{c(b-a)\xi_n s_n}$$

with some absolute constant $c>0$, and $n \geqslant n_0$. Therefore for any $x \in [a+\xi_n, b-\xi_n]$ such that $|T_{m_n}(x)| = 1$ and $n$ large enough

$$\left|\frac{\eta_n}{2} T_{m_n}(x)\, q_{2s_n}(x)\right| \geqslant \frac{\eta_n}{2} e^{c(b-a)\xi_n s_n} \geqslant \frac{1}{2} e^{-n\delta_n + c_1(b-a)n\delta_n^{5/6}} > 1.$$

Since $\|p_{m_n}\|_\infty = 1$ it follows that $Q_{m_n+2s_n}$ has at least $((\arccos(a+\xi_n) - \arccos(b-\xi_n))/\pi)\, m_n - O(1) = ((\arccos a - \arccos b)/\pi)n + o(n)$ zeros on $(a+\xi_n, b-\xi_n)$. On the other hand for $x \in [-1, 1] \backslash (a, b)$

$$\left|\frac{\eta_n}{2} T_{m_n}(x)\, q_{2s_n}(x)\right| \leqslant \frac{\eta_n}{2},$$

i.e., $Q_{m_n+2s_n}$ has at least $\tilde{N}(p_{m_n}, I \backslash [a, b]) - 2$ zeros in $I \backslash [a, b]$. Thus

$$\tilde{N}(p_{m_n}, I \backslash [a, b]) + \frac{\arccos a - \arccos b}{\pi} n + o(n) \leqslant m_n + 2s_n,$$

i.e.,

$$\limsup_{n \to \infty} \frac{1}{n} \tilde{N}(p_{m_n}, I \backslash [a, b]) \leqslant 1 - \frac{\arccos a - \arccos b}{\pi}$$

and using that $p_{m_n}$ has $n + o(n)$ oscillation on $I$

$$\liminf_{n \to \infty} \frac{1}{n} \tilde{N}(p_{m_n}, [a, b]) \geqslant \frac{\arccos a - \arccos b}{\pi}.$$

Since the last relation holds for every $-1 \leqslant a < b \leqslant 1$ the statement of the lemma follows easily. ∎

Now we are ready to prove Theorem 2.

*Proof of Theorem* 2. *Let* $g_{m_n} \in P_{m_n}$, $\|g_{m_n}\|_2 = 1$, $g_{m_n} \perp P_n$ $(n \in \mathbf{N})$ and $m_n/n \to 1$. Since $m_n = n + o(n)$ it follows by (3) that $\lim_{n \to \infty} \|g_{m_n}\|_\infty^{1/n} = 1$.

Hence by Lemma 4 for every $n \in \mathbf{N}$, $g_{m_n}$ possesses an oscillation of length $2n - m_n = n + o(n)$ and magnitude $\eta_n$ satisfying

$$1 = \lim_{n \to \infty} \left( \frac{\eta_n}{\| g_{m_n} \|_\infty} \right)^{1/n} = \lim_{n \to \infty} \eta_n^{1/n}. \tag{15}$$

Consider an arbitrary $[a, b] \subset [-1, 1]$ $(a < b)$. By Lemma 5 for $n$ sufficiently large, interval $[a, b]$ will contain at least one of the oscillation points, i.e.,

$$\max_{x \in [a,b]} |g_{m_n}(x)| \geqslant \eta_n \qquad (n \geqslant n_0).$$

Using this and inequality (3) (transformed to $[a, b]$) we have

$$1 \geqslant \int_a^b g_{m_n}^2 \omega \, dx \geqslant \eta_n^2 e^{-n\alpha_n},$$

where $\alpha_n = \alpha_n(a, b, \omega)$ converges to 0 as $n \to \infty$. This inequality and (15) easily imply (2).

Moreover, using again Lemma 5

$$\liminf_{n \to \infty} \frac{1}{n} N(g_{m_n}, [a, b]) \geqslant \frac{1}{\pi} (\arccos a - \arccos b)$$

for every $-1 \leqslant a < b \leqslant 1$, which implies (1).

By statement (2) of Theorem 2 if $g_{m_n} \perp P_n$, $\| g_{m_n} \|_2 = 1$, and $m_n/n \to 1$, then the $L_2$-norm of $g_{m_n}$ on $[a, b] \subset [-1, 1]$ can not tend to 0 geometrically. Moreover, as it was mentioned in the Introduction, geometric convergence to 0 is possible if $\lim_{n \to \infty}(m_n/n) > 1$. On the other hand, when $m_n = n + 1$ (i.e., $g_{m_n} = g_{n+1}$ is the orthonormal polynomial) we have the following stronger statement called sometimes Turán's inequality (see Turán [8] and Máté, Nevai, and Totik [5]),

$$\int_a^b g_{n+1}^2 \omega \, dx \geqslant \rho(b - a), \qquad n \in \mathbf{N}, \quad [a, b] \subset I, \tag{16}$$

where $\rho(b - a) > 0$ is a constant depending only on $b - a$ and $\omega$. Of course, in the special case when $m_n = n + 1$, (16) is essentially stronger than (2). It turns out that Turán's inequality can be extended to the case $m_n = n + r$, with $r$ being any fixed integer.

PROPOSITION. *Let $r \in \mathbf{N}$ and $\delta > 0$. Then for every $[a, b] \subset I$, with $b - a \geqslant \delta$, and every $g_{n+r} \in P_{n+r}$ such that $\| g_{n+r} \|_2 = 1$ and $g_{n+r} \perp P_n$ $(n \in \mathbf{N})$ we have*

$$\int_a^b g_{n+r}^2 \omega \, dx \geqslant \rho_{r,\omega}(\delta), \tag{17}$$

*where* $\rho_{r,\omega}(\delta) > 0$ *depends only on* $r$, $\omega$, *and* $\delta$.

*Proof.* Assume on the contrary that for some $\delta > 0$ there exist a sequence of polynomials $g_{n_k+r} \in P_{n_k+r}$ and intervals $E_k \subset I$ of length at least $\delta$ $(k \in \mathbf{N})$, such that $\|g_{n_k+r}\|_2 = 1$, $g_{n_k+r} \perp P_{n_k}$, and

$$\int_{E_k} g_{n_k+r}^2 \omega \, dx \to 0 \qquad (k \to \infty). \tag{18}$$

Without loss of generality we may assume that $E_k \supset [c, d]$ $(c < d)$ and $n_k \to \infty$ $(k \to \infty)$. (If $n_k \not\to \infty$ then a subsequence of $g_{n_k+r}$'s converges to a nontrivial polynomial, contradicting (18).) Evidently, $g_{n_k+r}$ can be written in the form

$$g_{n_k+r} = \sum_{j=1}^{r} a_{j,k} \, p_{n_k+j}, \tag{19}$$

where $p_s$ $(s \in \mathbf{N})$ stands for the orthonormal polynomial of degree $s$, and $a_{j,k} \in \mathbf{R}$ are such that $\sum_{j=1}^r a_{j,k}^2 = 1$. Since $|a_{j,k}| \leqslant 1$ $(1 \leqslant j \leqslant r,\ k \in \mathbf{N})$ we may assume that $a_{j,k} \to \bar{a}_j$ $(k \to \infty,\ 1 \leqslant j \leqslant r)$, where $\sum_{j=1}^r \bar{a}_j^2 = 1$. We have by (19)

$$\int_c^d g_{n_k+r}^2 \omega \, dx = \sum_{j=1}^{r} \sum_{s=1}^{r} a_{j,k} a_{s,k} \int_c^d p_{n_k+j} p_{n_k+s} \omega \, dx. \tag{20}$$

Furthermore, by [5, Theorem 11.1],

$$\lim_{k \to \infty} \int_c^d p_{n_k+j} p_{n_k+s} \omega \, dx = \frac{1}{\pi} \int_c^d T_{|j-s|} (1-x^2)^{-1/2} \, dx,$$

where as above $T_m(x) = \cos m \arccos x$ $(1 \leqslant s, j \leqslant r)$. Therefore, using (18), (20), and the above relation we have

$$\begin{aligned}
0 = \lim_{k \to \infty} \int_c^d g_{n_k+r}^2 \omega \, dx &= \sum_{j=1}^{r} \sum_{s=1}^{r} \bar{a}_j \bar{a}_s \frac{1}{\pi} \int_c^d T_{|j-s|} (1-x^2)^{-1/2} \, dx \\
&= \frac{1}{\pi} \sum_{j=1}^{r} \sum_{s=1}^{r} \bar{a}_j \bar{a}_s \int_{\bar{d}}^{\bar{c}} \cos(j-s)\varphi \, d\varphi \\
&= \frac{1}{\pi} \int_{\bar{d}}^{\bar{c}} \sum_{j=1}^{r} \sum_{s=1}^{r} \bar{a}_j \bar{a}_s \{\cos j\varphi \cos s\varphi + \sin j\varphi \sin s\varphi\} \, d\varphi \\
&= \frac{1}{\pi} \int_{\bar{d}}^{\bar{c}} \left( \sum_{j=1}^{r} \bar{a}_j \cos j\varphi \right)^2 + \left( \sum_{j=1}^{r} \bar{a}_j \sin j\varphi \right)^2 d\varphi.
\end{aligned}$$

But $\bar{d} = \arccos d < \arccos c = \bar{c}$ and $\sum_{j=1}^{r} \bar{a}_j^2 = 1$, and hence the last integral must be positive, a contradiction. ∎

Evidently, when $m_n = n + r$ with a fixed $r \in \mathbf{N}$ inequality (17) easily implies (2). On the other hand (17) can not be applied for verifying (2) when $m_n = n + r_n$ with $r_n = o(n)$ possibly unbounded (this is the setting needed for proving Theorem 1).

## Proof of Theorem 1

Assume that contrary to Theorem 1 there exist $f \in C[-1, 1]$, $[a, b] \subset [-1, 1]$, and a subsequence $\{n_j\}_{j=1}^{\infty}$ such that $\lim_{j \to \infty} n_{j+1}/n_j = 1$ and $f - B_{n_j, p}(f) \neq 0$ for every $j \in \mathbf{N}$ and $x \in [a, b]$ $(1 < p \leqslant \infty)$. We consider separately three cases: (A) $p = 2$; (B) $p = \infty$; (C) $1 < p < \infty$, $p \neq 2$. In principle, Case A could be imbedded into Case C, but we prefer to give a detailed proof of the simpler Case A, and then outline how the $L_p$-orthogonality

$$0 = \int_I g_n |f - B_{n,p}(f)|^{p-1} \operatorname{sgn}(f - B_{n,p}(f)) \omega \, dx, \qquad g_n \in P_n \tag{21}$$

can be applied in order to give a proof similar to Case A.

Set $a_j = E_{n_j, p}(f)$. Since $a_j \downarrow 0^+$ $(j \to \infty)$ it is known that

$$\sum_{j=1}^{\infty} \frac{a_j - a_{j+1}}{a_j + a_{j+1}} = \infty. \tag{22}$$

By Lemma 2.6 in [7], for every $n \in \mathbf{N}$ there exists $\bar{p}_n \in P_n$ such that $|\bar{p}_n| \leqslant 1$ for $x \in I \backslash [a, b]$; $\bar{p}_n \geqslant 0$ on $[a, b]$, and $\bar{p}_n \geqslant e^{\bar{c}n}$ on $[c, d] = [a + (b - a)/4, b - (b - a)/4]$ $(\bar{c} > 0$ is independent of $n)$.

*Case* A. $p = 2$. We may assume that $f - B_{n_j, 2}(f) > 0$ on $[a, b]$. Using (21) for $p = 2$, $n = n_j$, and $g_{n_j} = \bar{p}_{n_j}$ we have

$$\begin{aligned} a_j &\geqslant \int_I |f - B_{n_j, 2}(f)| \omega \, dx \geqslant \left| \int_{I \backslash [a,b]} \bar{p}_{n_j} (f - B_{n_j, 2}(f)) \omega \, dx \right| \\ &= \int_a^b \bar{p}_{n_j} (f - B_{n_j, 2}(f)) \omega \, dx \geqslant e^{\bar{c}n_j} \int_c^d |f - B_{n_j, 2}(f)| \omega \, dx. \end{aligned}$$

Analogously, applying (21) for $n = n_{j+1}$

$$a_{j+1} \geqslant e^{\bar{c}n_j} \int_c^d |f - B_{n_{j+1}, 2}(f)| \omega \, dx.$$

Setting $g_{n_{j+1}} = B_{n_{j+1},2}(f) - B_{n_j,2}(f) \in P_{n_{j+1}}$ and adding last two inequalities we obtain

$$a_j + a_{j+1} \geqslant e^{\bar{c}n_j} \int_c^d |g_{n_{j+1}}| \omega \, dx. \tag{23}$$

Using (23) and (3) (transformed to $[c, d]$) we have for some $\delta_j \to 0^+$ $(j \to \infty)$

$$a_j + a_{j+1} \geqslant e^{\bar{c}n_j - n_{j+1}\delta_j} \left( \int_c^d g_{n_{j+1}}^2 \omega \, dx \right)^{1/2}. \tag{24}$$

Since $n_{j+1}/n_j \to 1$ and $g_{n_{j+1}} \perp P_{n_j}$ it follows from Theorem 2 (relation (2)) that

$$\left( \int_c^d g_{n_{j+1}}^2 \omega \, dx \right)^{1/2} = e^{-n_j \varepsilon_j} \| g_{n_{j+1}} \|_2, \tag{25}$$

where $\varepsilon_j \to 0^+$ $(j \to \infty)$. Moreover, $\| g_{n_{j+1}} \|_2 \geqslant a_j - a_{j+1}$, thus combining (25) and (24) yields

$$a_j + a_{j+1} \geqslant e^{\bar{c}n_j - n_{j+1}\delta_j - n_j\varepsilon_j}(a_j - a_{j+1}),$$

i.e., for $j$ large enough and a proper $c_0 > 0$

$$\frac{a_j - a_{j+1}}{a_j + a_{j+1}} \leqslant e^{-c_0 n_j} \leqslant e^{-c_0 j}. \tag{26}$$

Evidently, (26) contradicts (22).

*Case* B. $p = \infty$. Let $x_i^{(j)}$, $1 \leqslant i \leqslant n_j + 2$, be points of equioscillation of $f - B_{n_j,\infty}(f)$, i.e.,

$$(f - B_{n_j,\infty}(f))(x_i^{(j)}) = \gamma(-1)^i a_j \qquad (|\gamma| = 1, 1 \leqslant i \leqslant n_j + 2). \tag{27}$$

Setting $p_{n_{j+1}} = B_{n_j,\infty}(f) - B_{n_{j+1},\infty}(f) \in P_{n_{j+1}}$ we have $\| p_{n_{j+1}} \|_\infty \leqslant a_j + a_{j+1}$, and, by (27), $\gamma(-1)^{i+1} p_{n_{j+1}}(x_i^{(j)}) \geqslant a_j - a_{j+1}$ $(1 \leqslant i \leqslant n_j + 2)$. Thus $p_{n_{j+1}}$ posesses an oscillation of length $n_j + 2$ and magnitude $\eta_j$ such that

$$1 \geqslant \eta_j / \| p_{n_{j+1}} \|_\infty \geqslant \frac{a_j - a_{j+1}}{a_j + a_{j+1}} \qquad (j \in \mathbf{N}).$$

Hence it follows from (22) that for some infinite set $\Omega \in \mathbf{N}$

$$\lim_{j \in \Omega, j \to \infty} (\eta_j / \| p_{n_{j+1}} \|_\infty)^{1/n_j} = 1.$$

Therefore Lemma 5 implies that points $x_i^{(j)}$, $1 \leqslant i \leqslant n_j + 2$, are uniformly distributed on $[-1, 1]$ for $j \in \Omega$. Since zeros of $f - B_{n_j,\infty}(f)$ interlace with the $x_i^{(j)}$'s it contradicts the assumption that $f - B_{n_j,\infty}(f) \neq 0$ in $[a, b]$.

*Case* C. $1 < p < \infty$, $p \neq 2$. Let us introduce some additional notations:

$$\begin{aligned}\Phi_j(x) &= |f - B_{n_j,p}(f)|^{p-1} \operatorname{sgn}(f - B_{n_j,p}(f)) \\ &\quad - |f - B_{n_{j+1},p}(f)|^{p-1} \operatorname{sgn}(f - B_{n_{j+1},p}(f)); \\ Q_{n_{j+1}}(x) &= B_{n_{j+1},p}(f) - B_{n_j,p}(f) \in P_{n_{j+1}}; \\ e_j(x) &= |f - B_{n_j,p}(f)| + |f - B_{n_{j+1},p}(f)|; \qquad a_j = E_{n_j,p}(f).\end{aligned}$$

By (21) we have

$$\int_I g_{n_j} \Phi_j \omega \, dx = 0, \qquad g_{n_j} \in P_{n_j}. \tag{28}$$

Furthermore, by [3, Proposition 2],

$$c_2(p) \begin{cases} e_j^{p-2} |Q_{n_{j+1}}|: & p<2 \\ |Q_{n_{j+1}}|^{p-1}: & p \geqslant 2 \end{cases} \leqslant |\Phi_j| \leqslant c_1(p) \begin{cases} |Q_{n_{j+1}}|^{p-1}: & p<2 \\ |Q_{n_{j+1}}|\, e_j^{p-2}: & p \geqslant 2 \end{cases} \tag{29}$$

$$\operatorname{sgn} \Phi_j = \operatorname{sgn} Q_{n_{j+1}}, \qquad x \in [-1, 1], \tag{30}$$

where $c_1(p)$, $c_2(p) > 0$ depend only on $p$.

Moreover, since $\sum_{j=1}^{\infty} ((a_j - a_{j+1})/(a_j + a_{j+1})) = \infty$, there exists an infinite subset $\Omega \subset \mathbf{N}$ so that

$$\frac{a_j - a_{j+1}}{a_j + a_{j+1}} \geqslant j^{-2}, \qquad j \in \Omega. \tag{31}$$

Consider now the polynomial $Q_{n_{j+1}}$, which by (30) and (28) has at least $n_j + 1$ sign changes in $[-1, 1]$. Applying Lemma 3 with $k = n_j + 1$, $n = n_j$, $m = n_{j+1}$ it follows that for some $g_{r_j} \in P_{r_j}$ (monic with $r_j$ simple zeros in $[-1, 1]$) with $n_{j+1} - n_j \leqslant r_j \leqslant n_{j+1} - n_j + 1$ and closed set $A_j \subset I$ we have $Q_{n_{j+1}}/g_{r_j} \in P_{n_{j+1} - r_j} \subseteq P_{n_j}$,

$$g_{r_j} \geqslant 0 \qquad \text{on} \quad I \backslash A_j, \tag{32}$$

and $Q_{n_{j+1}}$ has an oscillation on $I \backslash A_j$ of length $2n_j + 1 - n_{j+1} = n_j + o(n_j)$ and magnitude $\eta_j$ satisfying $\eta_j \geqslant \max_{x \in A_j} |Q_{n_{j+1}}(x)|$ ($j$ large enough).

Let us give a lower estimate for $\eta_j$. Assume at first that $p \geqslant 2$. We have by (28), (30), and (32)

$$0 = \int_I \frac{Q_{n_{j+1}}}{g_{r_j}} \Phi_j \omega \, dx = \int_{I \backslash A_j} \left| \frac{Q_{n_{j+1}}}{g_{r_j}} \Phi_j \right| \omega \, dx + \int_{A_j} \frac{Q_{n_{j+1}}}{g_{r_j}} \Phi_j \omega \, dx.$$

Thus by (29) and the Hölder inequality

$$c_2(p)\int_I \frac{|Q_{n_j+1}|^p}{|g_{r_j}|}\,\omega\,dx \leqslant \int_I \left|\frac{Q_{n_j+1}}{g_{r_j}}\Phi_j\right|\omega\,dx \leqslant 2\int_{A_j}\left|\frac{Q_{n_j+1}}{g_{r_j}}\Phi_j\right|\omega\,dx$$
$$\leqslant 2c_1(p)\int_{A_j}\frac{Q^2_{n_j+1}}{|g_{r_j}|}e_j^{p-2}\omega\,dx$$
$$\leqslant c_3(p)\,\eta_j\left\|\frac{Q_{n_j+1}}{g_{r_j}}\right\|_\infty (a_j+a_{j+1})^{p-2}.$$

On the other hand since $r_j=o(n_j)$ we have for the integral on the left side by Lemma 2

$$\int_I \frac{|Q_{n_j+1}|^p}{|g_{r_j}|}\,\omega\,dx = \int_I\left|\frac{Q_{n_j+1}}{g_{r_j}}\right|^p |g_{r_j}|^{p-1}\,\omega\,dx \geqslant e^{-n_j\varepsilon_j}\left\|\frac{Q_{n_j+1}}{g_{r_j}}\right\|^p_\infty,$$

where $\varepsilon_j\to 0^+$ $(j\to\infty)$. Combining the last two inequalities yields

$$\eta_j \geqslant c_4(p)\frac{e^{-n_j\varepsilon_j}}{(a_j+a_{j+1})^{p-2}}\left\|\frac{Q_{n_j+1}}{g_{r_j}}\right\|^{p-1}_\infty$$
$$\geqslant c_4(p)\frac{e^{-n_j\varepsilon_j}2^{-r_j(p-1)}}{(a_j+a_{j+1})^{p-2}}\|Q_{n_j+1}\|^{p-1}_\infty.$$

Thus using that $\|Q_{n_j+1}\|_\infty \geqslant a_j-a_{j+1}$ and $r_j=o(n_j)$ we have by (31) for $j\in\Omega$

$$\frac{\eta_j}{\|Q_{n_j+1}\|_\infty}\geqslant c_4(p)\left(\frac{a_j-a_{j+1}}{a_j+a_{j+1}}\right)^{p-2} e^{-n_j\varepsilon_j}2^{-r_j(p-1)}\geqslant e^{-n_j\delta_j}, \tag{33}$$

where $\delta_j\to 0^+$ $(j\to\infty,\ j\in\Omega)$.

Assume now that $1<p<2$. As above

$$c_2(p)\int_I\frac{Q^2_{n_j+1}}{|g_{r_j}|}e_j^{p-2}\,\omega\,dx$$
$$\leqslant\int_I\left|\frac{Q_{n_j+1}}{g_{r_j}}\Phi_j\right|\omega\,dx\leqslant 2\int_{A_j}\left|\frac{Q_{n_j+1}}{g_{r_j}}\Phi_j\right|\omega\,dx$$
$$\leqslant 2c_1(p)\int_{A_j}\frac{|Q_{n_j+1}|^p}{|g_{r_j}|}\,\omega\,dx\leqslant c_5(p)\,\eta_j^{p-1}\left\|\frac{Q_{n_j+1}}{g_{r_j}}\right\|_\infty.$$

Estimating the integral on the left side by the Hölder inequality in $L_q$, $0<q<1$, and using Lemma 2 we have

$$\int_I \frac{Q_{n_j+1}^2}{|g_{r_j}|} e_j^{p-2}\omega\,dx$$
$$\geqslant \left(\int_I \left|\frac{Q_{n_j+1}}{g_{r_j}}\right|^p |g_{r_j}|^{p/2}\omega\,dx\right)^{2/p} \cdot \left(\int_I e_j^p \omega\,dx\right)^{(p-2)/p}$$
$$\geqslant e^{-n_j\alpha_j} \left\|\frac{Q_{n_j+1}}{g_{r_j}}\right\|_\infty^2 (a_j+a_{j+1})^{p-2},$$

where $\alpha_j \to 0^+$ $(j\to\infty)$.

Combining the above inequalities we arrive at

$$\eta_j^{p-1} \geqslant c_6(p) e^{-n_j\alpha_j}(a_j+a_{j+1})^{p-2} \left\|\frac{Q_{n_j+1}}{g_{r_j}}\right\|_\infty$$
$$\geqslant c_6(p)\, 2^{-r_j} e^{-n_j\alpha_j}(a_j+a_{j+1})^{p-2} \|Q_{n_j+1}\|_\infty,$$

i.e., using again (31)

$$(\eta_j/\|Q_{n_j+1}\|_\infty)^{p-1} \geqslant c_6(p)\, 2^{-r_j} e^{-n_j\alpha_j} \left(\frac{a_j-a_{j+1}}{a_j+a_{j+1}}\right)^{2-p} \geqslant e^{-n_j\gamma_j}, \tag{34}$$

where $\gamma_j \to 0^+$ as $j\in\Omega$, $j\to\infty$. Thus by (33) and (34) for every $1<p<\infty$

$$\lim_{j\in\Omega,\, j\to\infty} (\eta_j/\|Q_{n_j+1}\|_\infty)^{1/n_j} = 1. \tag{35}$$

Recall that $Q_{n_j+1}$ possesses an oscillation of length $n_j+o(n_j)$ and magnitude $\eta_j$ satisfying (35). Since $n_{j+1}/n_j \to 1$ $(j\to\infty)$, it follows by Lemma 5 that these oscillation points are uniformly distributed on $[-1,1]$ for $j\in\Omega$. Thus for every $[c,d]\subset[-1,1]$ and $j\in\Omega$ big enough $\max\{|Q_{n_j+1}(x)|: x\in[c,d]\} \geqslant \eta_j$. Hence (35) and (5) (transformed to $[c,d]$) imply that for any $1<g<\infty$

$$\lim_{j\in\Omega,\, j\to\infty} \left\{\frac{\int_c^d |Q_{n_j+1}|^g \omega\,dx}{\|Q_{n_j+1}\|_\infty^g}\right\}^{1/n_j} = 1.$$

This relation replaces (2) when $p\neq 2$. Now the proof can be completed similarly to the case when $p=2$, using $L_p$-orthogonality (see [3] for details) instead of $L_2$-orthogonality. ∎

*Remark.* Note that in the case when $p=\infty$ we verified the following statement, which is somewhat stronger than Theorem 1: Let $\{n_j\}_{j=1}^{\infty}$, $n_1 < n_2 < \cdots$ be such that $n_{j+1}/n_j \to 1$, $j \to \infty$. Then for every $f \in C[-1, 1]$ there exists a subsequence $\Omega \subset \mathbf{N}$ such that zeros of $f - B_{n_j, \infty}(f)$, $j \in \Omega$, are uniformly distributed in $[-1, 1]$ (with respect to the Chebyshev measure).

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